How many coin tosses would you need until you get n Heads or m Tails?

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Experimental Math Seminar
–Joint work with Svante Janson
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Background and History

Problem of points

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The correspondence (snail mail) between Fermat and Pascal laid the foundation to what is now known as probability theory.

Problem of points (2)

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- Why?
- If player 1 has n rounds left to win and player 2 has m rounds to win, then we can translate that into tossing a fair coin until we get the corresponding face values.

Convention: For the rest of the talk, p represents the probability of getting Heads and $q := 1 - p$ represents the probability of getting Tails.

Probability generating function until reaching n Heads

Consider calculating the probability that we reach n Heads. Let $q = 1 - p$, then

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$n + t - 1$ is the maximum number of coin tosses and the binomial coefficient represents the number of ways to get $n - 1$ Heads out of $n + t - 1$ coin tosses.

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Remark: Fermat and Pascal were only interested in the probability of getting n Heads vs. getting m Tails.

Computations

Probability generating function (n Heads OR m Tails)

$X := X(n, m;\ p)$: The number of tosses until reaching (for the first time) either n Heads OR m Tails, where with probability p you get Heads.

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$$\begin{aligned} F(n, m;\ p)(x) = (px)^n \sum_{t=0}^{m-1} \binom{n+t-1}{n-1} (qx)^t \\ + (qx)^m \sum_{h=0}^{n-1} \binom{m+h-1}{m-1} (px)^h. \end{aligned}$$

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In the first sum: Each term $p^n \binom{n+t-1}{n-1} q^t$ represents the probability of getting n Heads and t Tails.

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Remark: $F(n, m;\ p)$ does not have a closed-form but *does* have a linear recurrence with polynomial coefficients. We do not show it here for sake of space.

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Remark: This becomes computationally slow. Using Wilf-Zeilberger algorithmic proof theory, we will obtain a third-order linear recurrence!

Results (1)

Recurrence using Wilf-Zeilberger theory

Let $L(n, m;\ p)$ be the expectation of $X \coloneqq X(n, m;\ p)$, which is the number of tosses until reaching (for the first time) either n Heads OR m Tails. Then,

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$$\begin{aligned} L(n, m;\ p) = & \frac{(pn + pm - 2p + 2n - 2)}{n-1} \cdot L\left(n-1, m\right) \\ & - \frac{(2pn + 2pm - 4p + n - 1)}{n-1} \cdot L\left(n-2, m\right) \\ & + \frac{p\left(m - 2 + n\right)}{n-1} \cdot L\left(n-3, m\right), \end{aligned}$$

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with initial conditions:

$$L(1,1)=1, L(1,2)=2-p, L(1,3)=p^2-3p+3, L(2,1)=p+1,$$
$$L(2,2)=-2p^2+2p+2, L(2,3)=3p^3-7p^2+3p+3$$
$$L(3,1)=p^2+p+1, L(3,2)=-3p^3+2p^2+2p+2,$$
$$L(3,3)=6p^4-12p^3+3p^2+3p+3.$$

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	Expected coin tosses
$n = 1$	3.272000000
$n = 2$	7.491773440
$n = 3$	11.90103592
$n = 4$	16.40588424
$n = 5$	20.97105153
$n = 6$	25.57874312
$n = 7$	30.21849553
$n = 8$	34.88353902
$n = 9$	39.56919552
$n = 10$	44.27207236

Table: The probability of getting Heads is $p = 2/5$.

Explicit Expression (an Heads OR bn Tails)

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$$L(an, bn,\ \tfrac{a}{a+b}) = (a+b)n\left(1 - \frac{((a+b)n)!}{(an)!(bn)!} \cdot \left(\frac{a^a b^b}{(a+b)^{a+b}}\right)^n\right).$$

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Let $a = 2, b = 3$ then $p = \frac{2}{5}$. If your goal is to flip the loaded coin until you reach $20,000$ Heads OR $30,000$ Tails, then

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the expected number of coin tosses is $50,182.09044$.

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One might be interested in the **asymptotics** of $L(an, bn,\ \frac{a}{a+b})$.

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Fix positive integers a, b. Let $p = \frac{a}{a+b}$ be the probability of getting Heads. If your goal is to reach either an Heads OR bn Tails, then $L(an, bn,\ \frac{a}{a+b})$ is asymptotically ($n \to \infty$),

$$(a+b)n\left(1 - \sqrt{\frac{a+b}{2ab\pi}} \cdot \frac{1}{\sqrt{n}}\right).$$

Pause

Summary so far!

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2. The expectation denoted by $L(n, m;\ p)$ of $X(n, m;\ p)$ satisfies a third-order linear recurrence.
3. If $p = \frac{a}{a+b}$, then we have given an explicit expression for $L(an, bn;\ p)$ for positive integers a, b.
4. Given asymptotics for $L(an, bn;\ p)$ with $p = \frac{a}{a+b}$.

Connection to other work

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Back to our language: You toss a coin until you get n Heads OR n Tails where with probability p you get Heads.

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Let $\mathbb{E}_{n,p}$ be the expectation of (the number of Heads - the number of Tails) after $X(n, n;\ p)$ tosses i.e. once we reach either n Heads OR n Tails. Then,

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$$\mathbb{E}_{n,p} = n(p-q)\sum_{j=0}^{n-1} C_j (pq)^j,$$

where $C_j = \frac{(2j)!}{j!(j+1)!}$ is the jth Catalan number.

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Their result is equivalent to our result in the special case of reaching n Heads OR n Tails.

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Let $X_1, X_2, \ldots$ be independent, identically distributed random variables with finite mean μ and let $S_n (n \geqslant 1)$, denote the partial sums. Suppose that τ is a stopping time. If $\mathbb{E}[\tau] < \infty$, then

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Takeaway:

total average profit= (average gain per toss)(average number of tosses)

Back to Volkov and Wiktorsson

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This is equivalent to

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In the fair case, $p = \frac{1}{2}$, we can verify that with $a = b = 1$, we get

$$L(n,n;\ \tfrac{1}{2}) = 2n\left(1 - \frac{(2n)!}{n!^2}4^{-n}\right).$$

Results (2)

Expected coin tosses figures for $n = m$

Using the recurrence for the expectation, we can compute the following with various p values.

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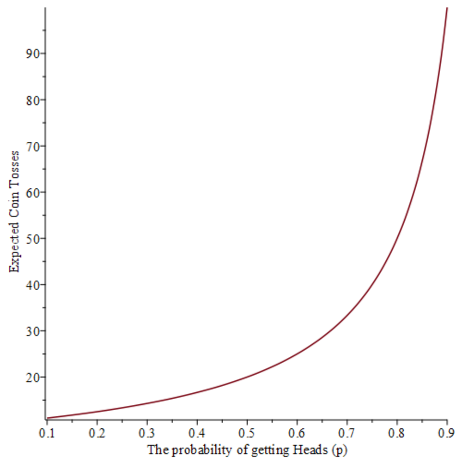
If your goal is to reach either 200 Heads OR 10 Tails,



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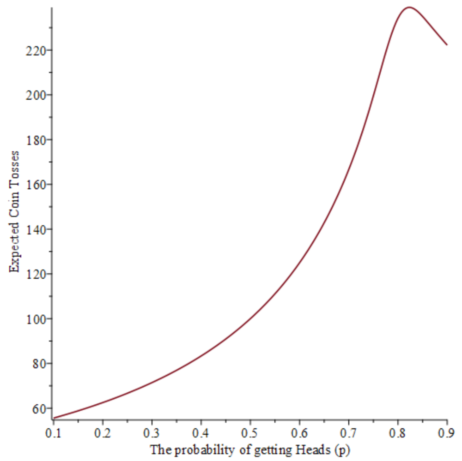
If your goal is to reach either 200 Heads OR 50 Tails,



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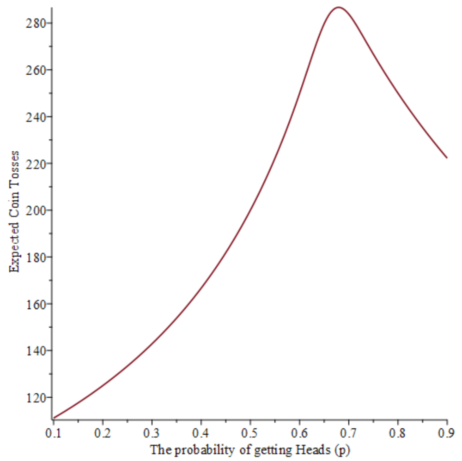
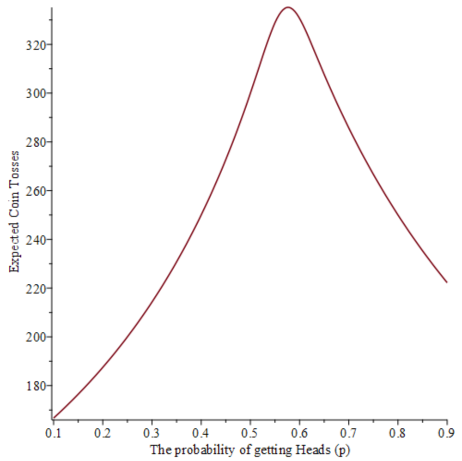
If your goal is to reach either 200 Heads OR 100 Tails,



Expected coin tosses figures for $n = m$

Using the recurrence for the expectation, we can compute the following with various p values.

If your goal is to reach either 200 Heads OR 150 Tails,



Expected coin tosses figures for $n = m$

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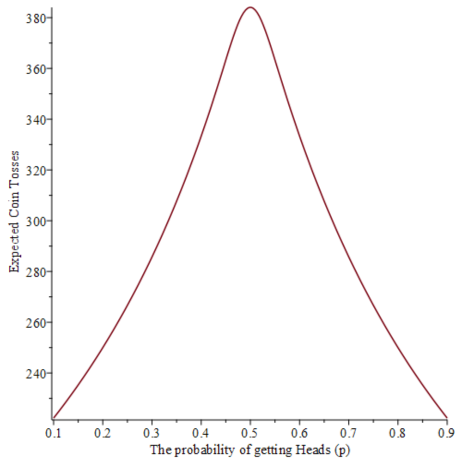

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$$A(n,1) = 2n - 2B_n, \quad A(n,2) = 4n^2 - 8nB_n, \quad B_n := \frac{n}{4^n}\binom{2n}{n}.$$

Central moments and scaled central moments

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Taking $n \to \infty$, we obtain the same scaled central moments of the **negative binomial distribution!**

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If you would like to read the paper:

`https://marti310.github.io/research.html`

Thank you!