A Mirror Step Variant of Gambler's Ruin

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Experimental Math Seminar

Outline

Background and History

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This scenario is known as the **gambler's ruin problem**, first posed by Pascal in 1656 in a letter to Fermat.

Questions

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- What is the probability of winning N dollars?

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Solution: $g(x) = x(N - x)$.

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Solution:

$$f(x) = \frac{1 - \left(\frac{q}{p}\right)^x}{1 - \left(\frac{q}{p}\right)^N}$$

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Restatement: A particle starts at a point x on a line segment of length N where $0 < x < N$. The particle moves to the left from x to $x - 1$ with probability $\frac{1}{2}$, or to the right from x to $x + 1$ with probability $\frac{1}{2}$.

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where $p + q_1 + q_2 = 1$.

We will focus on the case when $q_1 = q_2 = \frac{1-p}{2}$ (symmetric case).

Example

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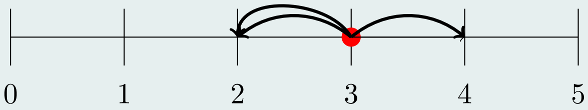

Results

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Expected Duration (2)

Example

Let $p = \frac{1}{2}$ then the recurrence is

$$g(x) = \frac{1}{4}g(x-1) + \frac{1}{4}g(x+1) + \frac{1}{2}g(N-x) + 1, \quad g(0) = 0, g(N) = 0.$$

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Let's compute some examples by varying N:

	Expected duration of ending at 0 or N starting at x				
N	$x = 1$	$x = 2$	$x = 3$	$x = 4$	$x = 5$
1	0				
2	2				
3	4	4			
4	6	8	6		
5	8	12	12	8	
6	10	16	18	16	10

Expected Duration (3)

	Expected duration of ending at 0 or N starting at x				
N	$x = 1$	$x = 2$	$x = 3$	$x = 4$	$x = 5$
1	0				
2	1				
3	2	2			
4	3	4	3		
5	4	6	6	4	
6	5	8	9	8	5

	Expected duration of ending at 0 or N starting at x when $p = 1/2$				
N	$x = 1$	$x = 2$	$x = 3$	$x = 4$	$x = 5$
1	0				
2	2				
3	4	4			
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Expected Duration

Theorem (1)

Consider the generalization of the gambler's ruin problem when we add a mirror step. Then, the expected duration of ending at 0 *or* N *starting at* x *is given by*

$$g(x) = \frac{1}{1-p}x(N-x)$$

whenever we restrict the particle moves by either moving from x *to* $x-1$ *with probability* q_1*, or from* x *to* $x+1$ *with probability* q_2*, or jumps to* $N-x$ *with probability* p *where* $q_1 = q_2 = \frac{1-p}{2}$*.*

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Remark: When $p = 0$, Theorem (1) recovers the formula for the expected duration of the classical gambler's ruin game.

Probability

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Define $f(x) = f_N^{(p)}(x)$ as the probability that a particle starting at x will eventually reach N. For $0 < x < N$, this probability satisfies the recurrence relation

$$f(x) = \frac{1-p}{2} f(x-1) + \frac{1-p}{2} f(x+1) + pf(N-x)$$

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Let's compute some examples by varying N:

	Probability of ending at N starting at x				
N	$x=1$	$x=2$	$x=3$	$x=4$	$x=5$
1	1				
2	1/2				
3	3/7	4/7			
4	5/12	1/2	7/12		
5	17/41	20/41	21/41	24/41	
6	29/70	17/35	1/2	18/35	41/70

Example (continued)

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4	1/4	1/2	3/4		
5	1/5	2/5	3/5	4/5	
6	1/6	1/3	1/2	2/3	5/6

	Probability of ending at N starting at x when $p = 1/2$				
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We will try to guess the limit as $N \to \infty$ of the probability when the particle starts at some x and ends at N. We are interested in

$$\lim_{N\to\infty} f_N^{(p)}(x)$$

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Let $N = 100, x = 1$ and set $p = \frac{1}{k}$ for $k \in \{2, 3, \ldots, 9\}$.

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p	$\lim_{N\to\infty} f_{100}^{(p)}(1)$
1/2	0.4142135624
1/3	0.3660254038
1/4	0.3333333333
1/5	0.3090169944
1/6	0.2898979486
1/7	0.2742918852
1/8	0.2612038750
1/9	0.2500000000

Experiments (2)

Maple contains a function called *identify* which is based, in part, on the continued fraction expansion of any given numerical value. Let's use that function on our data:

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1/2	$\sqrt{2}-1$
1/3	$\frac{\sqrt{3}-1}{2}$
1/4	$\frac{1}{3}$
1/5	$\frac{\sqrt{5}-1}{4}$
1/6	$\frac{\sqrt{6}-1}{5}$
1/7	$\frac{\sqrt{7}-1}{6}$
1/8	$\frac{2\sqrt{2}-1}{7}$
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Experiments (3)

What does this suggest?

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It suggests that the probability of the particle starting at $x = 1$ and ending at 100 converges to some number!

Guess (1)

Guess ($x = 1$)

If the particle starts at $x = 1$, then

$$\lim_{N \to \infty} f_N^{(p)}(1) = \frac{\sqrt{p} - p}{1 - p}.$$

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From previous slide,

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More Guesses (2)

Guess ($x = 2$)

If the particle starts at $x = 2$, then

$$\lim_{N \to \infty} f_N^{(p)}(2) = \frac{2\sqrt{p}(1 + p - 2\sqrt{p})}{(1-p)^2}.$$

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$$\lim_{N\to\infty} f_N^{(p)}(N-2) = \frac{(1+p)(1+p-2\sqrt{p})}{(1-p)^2}.$$

Check that the sum (of the numerators) equals $(p-1)^2$.

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Let us look at squared p values:

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$1/2^2$	$\frac{1}{3}$
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We can conjecture that $\lim_{N\to\infty} f_N^{(p)}(1) = \frac{1}{k+1}$ but $k = \frac{1}{\sqrt{p}}$ and $p = n^2$.
Together,

$$\lim_{N\to\infty} f_N^{(p)}(1) = \frac{n}{n+1}$$

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$x = 1$	$\frac{n}{n+1}$		$\frac{1}{n+1}$	$x = N - 1$
$x = 2$	$\frac{2n}{(n+1)^2}$		$\frac{n^2+1}{(n+1)^2}$	$x = N - 2$
$x = 3$	$\frac{n^3+3n}{(n+1)^3}$		$\frac{3n^2+1}{(n+1)^3}$	$x = N - 3$
$x = 4$	$\frac{4n^3+4n}{(n+1)^4}$		$\frac{n^4+6n^2+1}{(n+1)^4}$	$x = N - 4$

Binomial Theorem

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We can take the odd (or even) part:

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Check: Plug in $k = 3$ and $x = n$

$$\frac{1}{(n+1)^3} \sum_{i=0}^{1} \binom{3}{2i} n^{3-2i} = \frac{n^3 + 3n}{(n+1)^3}$$

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$x = 1$	$\frac{n}{n+1}$
$x = 3$	$\frac{n^3+3n}{(n+1)^3}$

$\frac{1}{n+1}$	$x = N - 1$
$\frac{3n^2+1}{(n+1)^3}$	$x = N - 3$

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$x = 1$	$\frac{n}{n+1}$
$x = 3$	$\frac{n^3+3n}{(n+1)^3}$

$\frac{1}{n+1}$	$x = N - 1$
$\frac{3n^2+1}{(n+1)^3}$	$x = N - 3$

Turns out the odd part yields the same closed form from above.

Result

Corollary

If the particle starts at some x where $0 < x < N$, then

$$\lim_{N\to\infty} f_N^{(p)}(x) = \frac{1}{2} - \frac{1}{2}\left(\frac{1-\sqrt{p}}{1+\sqrt{p}}\right)^x$$

whenever we restrict the particle moves by either moving from x to $x-1$ with probability q_1, or from x to $x+1$ with probability q_2, or from x to $N-x$ with probability p where $q_1 = q_2 = \frac{1-p}{2}$.

Miracle

Lemma

Consider the symmetric case when

$$f(x) = \frac{1-p}{2} f(x-1) + \frac{1-p}{2} f(x+1) + pf(N-x)$$

with boundary conditions $f(0) = 0, f(N) = 1$ *for some* $0 < p < 1$. *For any* $0 \leq x \leq N$, *the following identity holds*

$$f(x) + f(N-x) = 1.$$

Proof

Lemma

For any $0 \leq x \leq N$, *the following identity holds*

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Proof (Sketch).

Consider $f(x) = 1 - f(N - x)$. Then,
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The distance of both random walks is $N - x$.
Hence, $f(x) = 1 - f(N - x)$. □

Recurrence Relation

Since $f(N-x) = 1 - f(x)$, then we can rewrite the recurrence relation as follows

$$f(x) = \frac{p}{1+p} + \frac{1}{2}\left(\frac{1-p}{1+p}\right) f(x-1) + \frac{1}{2}\left(\frac{1-p}{1+p}\right) f(x+1)$$

where $f(0) = 0, f(N) = 1$.

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Consider the generalization of the gambler's ruin problem when we add a mirror step. If the particle starts at $x = 1$, then

$$\lim_{N\to\infty} f_N^{(p)}(1) = \frac{\sqrt{(p+1)(1-3p+4p^2)} - (1-2p)(p+1)}{2p(p+1)}$$

whenever we restrict the particle moves by either moving from x to $x-1$ with probability q_1, or from x to $x+1$ with probability p, or from x to $N-x$ with probability q_2 where $q_1 = q_2 = \frac{1-p}{2}$.

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Thank You!

https://marti310.github.io/research.html

Probability Formula

Theorem

Consider the generalization of the gambler's ruin problem when we add a mirror step. Then, the probability of ending at N starting at x is given by

$$f(x) = \frac{1}{2}\frac{\left(\frac{1-\sqrt{p}}{1+\sqrt{p}}\right)^N + 1}{\left(\frac{1+\sqrt{p}}{1-\sqrt{p}}\right)^N - \left(\frac{1-\sqrt{p}}{1+\sqrt{p}}\right)^N}\left(\frac{1+\sqrt{p}}{1-\sqrt{p}}\right)^x + \frac{1}{2}\frac{\left(\frac{1+\sqrt{p}}{1-\sqrt{p}}\right)^N + 1}{\left(\frac{1-\sqrt{p}}{1+\sqrt{p}}\right)^N - \left(\frac{1+\sqrt{p}}{1-\sqrt{p}}\right)^N}\left(\frac{1-\sqrt{p}}{1+\sqrt{p}}\right)^x + \frac{1}{2}$$