The Power of Computation

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Rutgers University

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Advisor: Dr. Doron Zeilberger

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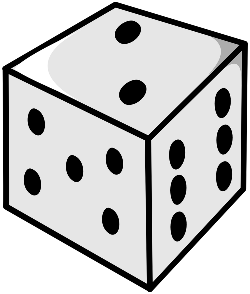
Game 1: Dice Problem

Dice Rolls



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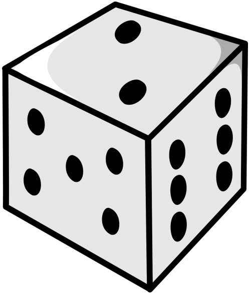
Dice Rolls



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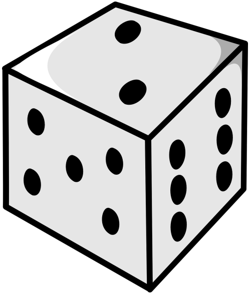

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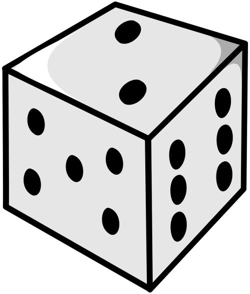
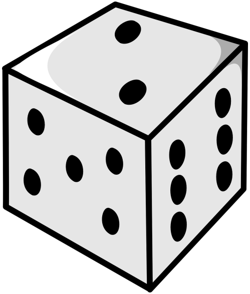


Suppose you have a fair standard die:

- Faces: $\{1, 2, 3, 4, 5, 6\}$
- Set $s = 0$. Roll the die and add the outcome to s
- Keep rolling the die, add the outcome to s, and stop when s is a prime number

Example



Set $s = 0$, and let's roll a die:

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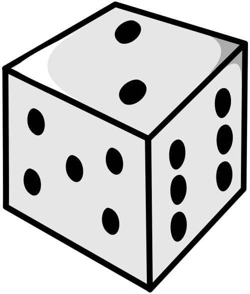

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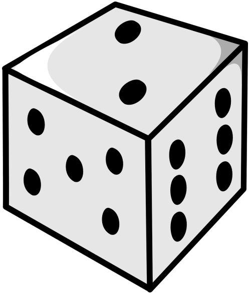

Set $s = 0$, and let's roll a die:

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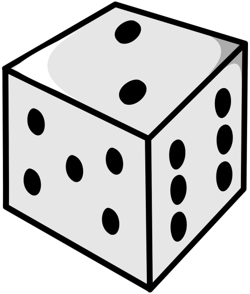

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- Roll again: say we get a 3, then $s = 10 + 3 = 13$ (prime!)

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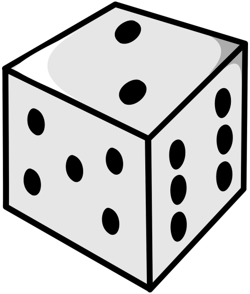

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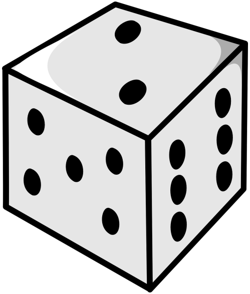
Total rolls: 3

Example 2



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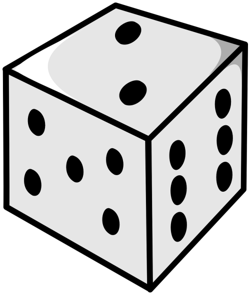

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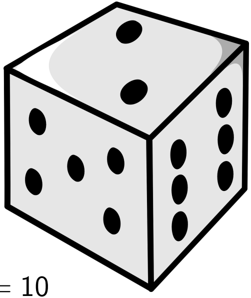

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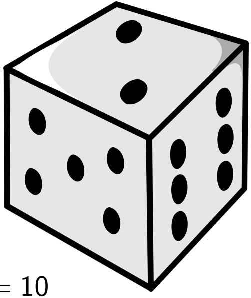

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- Roll again: say we get a 4, then $s = 10 + 4 = 14$

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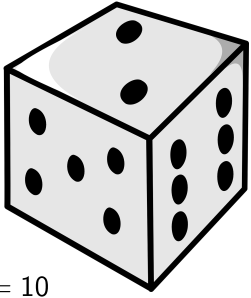

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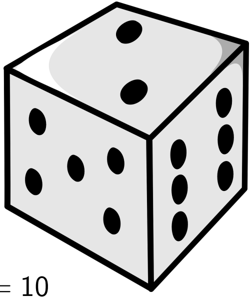

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- We would not be able to get a prime sum.

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- First roll: $\{2, 3, 5\}$, and with probability $1/2$ the game lasts one round

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  The probability that the game lasts 2 rounds is $1/9 + 1/18 + 1/18 = 2/9$

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With probability $1 - 1/2 - 2/9 = 5/18$, you need to continue

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Alon-Malinovsky (2022)

The expectation of this random variable (up to an additive error of less than 10^{-4}) is 2.484

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- Turns out: $E_{1000} = 2.4284$ is a good approximation (Alon-Malinovsky)

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- What if instead of a standard die with six faces, you have a different number of faces?
- What if instead of trying to hit a prime, you want to hit your favorite numbers? Say a product of two distinct primes, product of three distinct primes, perfect square (starting at a non-square), etc.

Our Approach: Symbolic Computation

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We used Maple to implement this function.

Symbolic Computation (continued)

- To compute $F_R(t, x)$: Define
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First Two Rounds

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$$S_0(x) = 1$$
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$$\begin{aligned} S_0(x) &= 1 \\ N_1(x) &= \mathcal{P}(P(x)S_0(x)) \\ &= \mathcal{P}\left(\frac{1}{6}(x + x^2 + x^3 + x^4 + x^5 + x^6) \cdot 1\right) \end{aligned}$$

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\end{aligned}
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&= \frac{1}{6}(x^2 + x^3 + x^5) \\
S_1(x) &= P(x)S_0(x) - N_1(x) = \frac{1}{6}(x + x^4 + x^6)
\end{aligned}
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S_1(x) &= P(x)S_0(x) - N_1(x) = \frac{1}{6}(x + x^4 + x^6) \\
\Longrightarrow \quad F_1(t, x) &= \left(\frac{1}{6}(x^2 + x^3 + x^5)\right) t
\end{aligned}
$$

First Two Rounds (continued)

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&= \frac{1}{36}x^2 + \frac{1}{36}x^3 + \frac{1}{18}x^5 + \frac{1}{12}x^7 + \frac{1}{36}x^{11}
\end{aligned}
$$

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&= \frac{1}{36}x^2 + \frac{1}{36}x^3 + \frac{1}{18}x^5 + \frac{1}{12}x^7 + \frac{1}{36}x^{11} \\
S_2(x) &= \frac{1}{36}x^4 + \frac{1}{18}x^6 + \frac{1}{18}x^8 + + \frac{1}{18}x^9 + \frac{1}{18}x^{10} + \frac{1}{36}x^{12}
\end{aligned}
$$

First Two Rounds (continued)

- Second Round

$$
\begin{aligned}
N_2(x) &= \mathcal{P}(P(x)S_1(x)) \\
&= \mathcal{P}\left(\frac{1}{6}(x + x^2 + x^3 + x^4 + x^5 + x^6) \cdot \frac{1}{6}(x + x^4 + x^6)\right) \\
&= \frac{1}{36}x^2 + \frac{1}{36}x^3 + \frac{1}{18}x^5 + \frac{1}{12}x^7 + \frac{1}{36}x^{11} \\
S_2(x) &= \frac{1}{36}x^4 + \frac{1}{18}x^6 + \frac{1}{18}x^8 + + \frac{1}{18}x^9 + \frac{1}{18}x^{10} + \frac{1}{36}x^{12} \\
\Longrightarrow F_2(t, x) &= \left(\frac{1}{6}(x^2 + x^3 + x^5)\right) t \\
&+ \left(\frac{1}{36}x^2 + \frac{1}{36}x^3 + \frac{1}{18}x^5 + \frac{1}{12}x^7 + \frac{1}{36}x^{11}\right) t^2
\end{aligned}
$$

- First roll: $\{2, 3, 5\}$, and with probability $1/2$ the game lasts one round
- Second roll:

  Possible outcomes in the first round: $\{1, 4, 6\}$

  How can we get a prime sum if $s = 1, s = 4$ or $s = 6$?

  - If $s = 1$, we must roll: $\{1, 2, 4, 6\}$. Probability to get a prime sum is $1/6 \cdot 4/6 = 1/9$
  - If $s = 4$, we must roll: $\{1, 3\}$. Probability to get a prime sum is $1/6 \cdot 2/6 = 1/18$
  - If $s = 6$, we must roll: $\{1, 5\}$. Probability to get a prime sum is $1/6 \cdot 2/6 = 1/18$

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Note: The coefficient of $F_2(t,x)$ at t^2 was

$$\frac{1}{36}x^2 + \frac{1}{36}x^3 + \frac{1}{18}x^5 + \frac{1}{12}x^7 + \frac{1}{36}x^{11}$$

Non-rigorous Estimates - Results

Number of Faces	Property	Expected Duration
7	prime sum	$2.1364\cdots$
12	prime sum	$3.0814\cdots$
6	product of two distinct primes	$3.7889\cdots$
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Note: To find the expected duration, we compute the partial derivative with respect to t of $F_R(t,x)$, evaluate at $t = x = 1$, and then divide by $F_R(1,1)$.

Game 2: St. Petersburg Paradox

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We would want to pay any amount A since $\infty - A = \infty$.

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In this case, if the gambler pays any amount A, then to ensure they do not lose money, $A < k + 1$.

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Example

Say $n = 6$ and $A = 0$. Then, StPetePT(6,0) outputs

$$[[2, 1/2], [4, 1/4], [8, 1/8], [16, 1/16], [32, 1/32], [32, 1/32]]$$

Simulation (continued)

Next, we simulate the game in Maple:

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Let
$M = [[2, 1/2], [4, 1/4], [8, 1/8], [16, 1/16], [32, 1/32], [32, 1/32]]$,
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Spoiler Alert: Using symbolic computation, the exact probability is $0.9088\cdots$

Symbolic Computation

Let $M = [[M_1, p_1], [M_2, p_2], \cdots, [M_r, p_r]]$. Assume $M_1, \cdots, M_r$ are integers.

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we get

$$P_M(x) = \frac{1}{2}x^2 + \frac{1}{4}x^4 + \frac{1}{8}x^8 + \frac{1}{16}x^{16} + \frac{1}{16}x^{32}$$

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we get

$$P(x)^+ = \frac{1}{8} + \frac{1}{16} + \frac{1}{16} = 1/4$$

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Example

Playing the following

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for $n = 100$ times, we get the exact probability of 0.9088286275.

Advanced Computation

Essentially, we are interested in calculating

$$(P_M(x)^n)^+ = \sum_{j=1}^{\infty} \text{Coeff}_{x^j}(P_M(x)) = \cdots = \frac{1}{2\pi i} \int_{|x|=1} \frac{(P_M(x))^n}{x(x-1)} dx$$

for $n \in \mathbb{N}$, where $\text{Coeff}_{x^j}(P_M(x))$ is the coefficient of x_j in $P_M(x)$.

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Using this *theorem*, we can get a good approximation for sufficiently large n.

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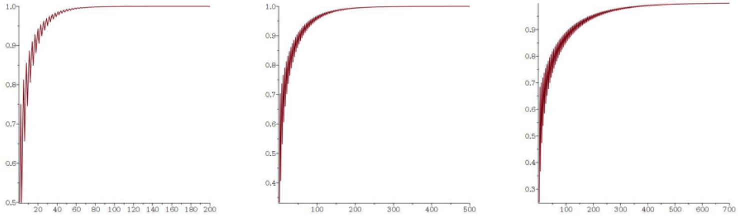
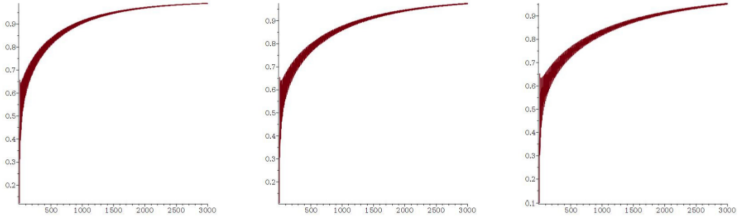


(a) $[[-1, 1/2], [2, 1/2]]$ (b) $[[-1, 2/3], [3, 1/3]]$ (c) $[[-1, 3/4], [4, 1/4]]$



(d) $[[-1, 7/8], [8, 1/8]]$ (e) $[[-1, 8/9], [9, 1/9]]$ (f) $[[-1, 9/10], [10, 1/10]]$

Figure 1. The risk-averseness graphs for the corresponding gambles.

Conclusion

- Dice Game and St. Petersburg Paradox

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- Dice Game and St. Petersburg Paradox
- Simulation, and Symbolic Computation

Thank You!

Resources I

Lucy Martinez and Doron Zeilberger.
How many dice rolls would it take to reach your favorite kind of number?
To appear in Maple Transactions, 2023.

Lucy Martinez and Doron Zeilberger.
A guide to the risk-averse gambler and resolving the st. petersburg paradox once and for all.

Noga Alon and Yaakov Malinovsky.
Hitting a prime in 2.43 dice rolls (on average).
The American Statistician, 2023.

Numerical Dynamic Programming

Alon-Malinovsky

For $k \leq n \leq 6k$ where n is non-prime, define $p(n, k)$ to be the probability that after k rolls, the running sum is n. Then,

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where $i \in \{1, 2, \cdots, 6\}$ such that $n - i$ is non-prime.

Observe that $p(1, 1) = p(1, 4) = p(1, 6) = 1/6$.

Building Intuition

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$$E(\tau) = \sum_{k \geq 1} p(k) = p(1) + p(2) + \cdots$$

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It turns out,

$$p(k+1) := \sum_{n:\ k \leq\ n\ \leq 6k} p(k, n), \text{ non-prime } n$$

Why?

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Why? For non-prime n:

$$\begin{aligned} \boldsymbol{p}(\mathbf{3}) &= \sum_{3 \leq n \leq 18} p(2, n) \\ &= p(2,4) + p(2,6) + \cdots + p(2,18) \end{aligned}$$